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Singular Behavior of the Density of States and the Lyapunov Coefficient in Binary Random Harmonic Chains

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We study the integrated density of states $H(\omega^2)$ of a chain of harmonic oscillators with a binary random distribution of the masses. We show in particular that there is a dense set of values of the squared frequency for which the difference $H(\omega^2+\varepsilon)-H(\omega^2)$ has a singularity of the type $|\varepsilon|^{2\alpha}$, multiplied by a periodic function of $\ln|\varepsilon|$, where the exponent α and the period depend continuously on ω^2. In the region where $\alpha<1/2$, H is not differentiable on a dense set of points. The same type of singularities is also present in the Lyapunov coefficient.

KEY WORDS: Density of states; random harmonic chains; one-dimensional systems; dense sets of singularities; Lyapunov coefficient.

1. INTRODUCTION

The problem of calculating the density of states of a chain of coupled harmonic oscillators with random masses has been studied for a long time. The first theoretical description was given by Dyson,[1] who showed that, for imaginary frequencies, the problem may be reduced to solving an integral equation for the distribution function of certain continued fractions. Schmidt[2] derived a similar integral equation for real frequencies. The relation between these two approaches has recently been discussed by one of the authors.[3]

Schmidt noted a peculiarity of his distribution function in the case of a *binary* mass distribution: under some circumstances the derivative of this

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function, if it exists, is zero or infinite on a dense set of points. This observation will be the cornerstone of our investigations.

It was then found numerically[(4)] that the density of states has a very irregular behavior at high frequency, if the ratio of heavy and light masses is greater than or equal to 2. In related tight-binding electron models[(5,6)] with a binary distribution of site energies, the density of states was also shown to possess detailed structure down to a very small scale. The authors of Ref. 5 concluded that a continuous density of states might not exist. A real-space renormalization group approach has recently been considered by various authors.[(7–10)] Their numerical or approximate results also show very detailed structure in the density of states, which again might rise the question whether this function is a well-defined quantity.

A simple physical argument predicting power-law singularities in the integrated density of states of related electronic models was given by Halperin[(33)] (see also Section 2). In the language of harmonic chains, his result is that, near a dense set of frequencies ω_0^2, the integrated density of states $H(\omega^2)$ admits the bound:

$$H(\omega_0^2+\varepsilon)-H(\omega_0^2-\varepsilon)>C\varepsilon^{2\alpha}$$

for $\varepsilon\to 0$, where C is some constant independent of ε. In the present paper, we shall refine this result and derive the following behavior:

$$H(\omega_0^2\pm\varepsilon)-H(\omega_0^2)\approx \pm\varepsilon^{2\alpha}R_\pm\left(\frac{\ln\varepsilon}{\ln\mu}\right) \tag{1.1}$$

Here $R_\pm$ are two positive periodic functions with unit period, which we relate to the Schmidt function *at* ω_0^2. In other words, the behavior of H in a small frequency interval around ω_0^2 is expressed in terms of the Schmidt function at one single frequency ω_0^2. The exponent α and the scale μ both depend continuously on ω_0^2. These singular points are generically *dense* in the interval $4/M\leqslant\omega^2\leqslant 4$ where we assume that the masses only take the values 1 or M. Under certain conditions, the index α is less than 1/2 in a high-frequency region. Then the density of states $dH/d\omega^2$ diverges on a dense set of points.

The setup of this paper is as follows. In Section 2 we recall some basic definitions and notation. We also recall the argument of Halperin.[(33)] Section 3 is devoted to the Schmidt distribution function $W(u)$, which is singular on a dense set of the real line; these power-law singularities are multiplied by periodic functions. It is shown in Section 4 that the integrated density of states $H(\omega^2)$ exhibits the very same type of singularities, as mentioned in Eq. (1.1). The Lifshitz (exponential) singularity at $\omega^2=4$ is studied in Section 5: it is also modulated by a periodic amplitude. In Section 6, we generalize our results to arbitrary *discrete* mass distributions. We also show that the Lyapunov coefficient $\gamma(\omega^2)$ has an analogous dense set of power-law singularities, and present some concluding remarks.

2. THE MODEL

Consider an ensemble of random harmonic chains described by the equations of motion:

$$-m_n\omega^2 a_n = a_{n+1} + a_{n-1} - 2a_n \qquad \left(-\frac{N}{2} \leqslant n \leqslant \frac{N}{2}\right) \tag{2.1}$$

where the masses m_n are independent, identically distributed random variables with common distribution function $R(m)$. We choose the following boundary conditions: $a_{-N/2-1} = a_{N/2+1} = 0$, and the normalization condition $a_{-N/2} = 1$. Equation (2.1) is then equivalent to the following continued fraction expression:

$$Z_n = \frac{a_n}{a_{n+1}} = (2 - m_n\omega^2 - Z_{n-1})^{-1} = T_{m_n}(Z_{n-1}) \tag{2.2}$$

with the initial value $Z_{-N/2-1} = 0$. Eigenfrequencies have to satisfy $Z_{N/2} = \infty$.

The Z_n are random variables since they are functions of the m_l ($l \leqslant n$). Let us define their distribution functions at fixed frequency ω^2 through

$$W_n(u; \omega^2) = \text{prob}\{Z_n \leqslant u\} - \text{prob}\{Z_n \leqslant 0\} \tag{2.3}$$

We shall not mention ω^2 explicitly in unambiguous equations.

The recursion equation (2.2) implies the following relation between W_n and W_{n-1}:

$$W_n(u) = \int dR(m)\, W_{n-1}(2 - m\omega^2 - u^{-1}) - \Theta(-u) - W_{n-1}(-\infty) \tag{2.4}$$

where Θ is Heaviside's step function.

The existence of a limit $W(u) = \lim_{n\to\infty} W_n(u)$ was proven by Furstenberg,[34] and later, unaware of this point of Furstenberg's work, for distributions $R(m)$ with bounded densities by Verheggen,[11] and then for arbitrary distributions by Nieuwenhuizen.[3] However Schmidt[2] circumvented this difficulty by considering the averages:

$$W^{(N)}(u) = \frac{1}{N+1} \sum_{-N/2 \leqslant n \leqslant N/2} W_n(u) \tag{2.5}$$

which converge to the same limit function $W(u)$. It satisfies the integral equation:

$$W(u) = \int dR(m)\, W(2 - m\omega^2 - u^{-1}) - \Theta(-u) - W(-\infty) \tag{2.6}$$

The integrated density of states $H(\omega^2)$ is related to $W(u; \omega^2)$. Indeed, for a large but finite chain, $H(\omega^2)$ is approximately equal to the number of nodes in the sequence a_n $(-N/2 \leqslant n \leqslant N/2)$, divided by N. In terms of the ratios Z_n, $H(\omega^2)$ is just the fraction of negative Z_n. One therefore has with probability 1 (or after averaging over the ensemble)[2]

$$H(\omega^2) = \text{prob}\{Z < 0\} = -W(-\infty; \omega^2) \tag{2.7}$$

The Lyapunov coefficient, or inverse localization length, $\gamma(\omega^2)$ is also related to $H(\omega^2)$ and $W(u; \omega^2)$. This quantity is defined as

$$\gamma(\omega^2) = \lim_{N \to \infty} \frac{1}{N} \langle \ln |a_N| \rangle \tag{2.8}$$

where a_N is the solution of Eq. (2.1) with boundary conditions $a_0 = 0$; $a_1 = 1$. γ and H can be related by the Herbert–Jones–Thouless formula:

$$\gamma(\omega^2) = \int dH(\omega'^2) \ln \left| 1 - \frac{\omega^2}{\omega'^2} \right| \tag{2.9}$$

In Section 6, we shall use the identity

$$\gamma(\omega^2) = -\int dW(u; \omega^2) \ln |u| \tag{2.10}$$

which can easily deduced from Eqs. (2.2), (2.3), (2.8).

We now briefly recall the argument of Halperin.[33] Consider a chain where the masses can only take two values, namely, $m_n = 1$ with probability p and $m_n = M > 1$ with probability $(1-p)$. Let ω_0^2 be the eigenfrequency of one light particle in a sea of heavy ones [see Eq. (3.12)]. It turns our that ω_0^2 exceeds the highest frequency of a chain consisting only of heavy masses, so that they act as damping centers for the mode at ω_0^2. Now if there are only N heavy masses, this eigenfrequency will be shifted over an amount of order μ^{-N}, where μ is some fixed number. The probability for occurrence of such a situation is $p(1-p)^{2N}$. The integrated density of states between ω_0^2 and $\omega^2 = \omega_0^2 \pm \mu^{-N}$ is therefore approximately equal to

$$|H(\omega^2) - H(\omega_0^2)| \sim \sum_{n \geqslant N} p(1-p)^{2n} \sim (1-p)^{2N}$$

Eliminating N, one recovers the power-law behavior (1.1), with $\alpha = -\ln(1-p)/\ln \mu$. See Eqs. (3.3) and (3.8) for the explicit value of μ. Although this argument is very simple, it is already completely rigorous (see Ref. 27).

3. SINGULAR BEHAVIOR OF THE SCHMIDT FUNCTION

Assume that the masses m_n can take only two values: $m_n = 1$ with probability p, and $m_n = M > 1$ with probability $(1-p)$. Then the Schmidt equation (2.6) reads

$$W(u) = pW(2-\omega^2-u^{-1}) + (1-p)\, W(2-M\omega^2-u^{-1}) \\ - \Theta(-u) - W(-\infty) \tag{3.1}$$

Let us now reproduce Schmidt's argument on the "strange" behavior of the density $W'(u)$. If $W'(u)$ exists, then it satisfies $(u \neq 0)$

$$W'(u) = pu^{-2}W'(2-\omega^2-u^{-1}) + (1-p)\, u^{-2}W'(2-M\omega^2-u^{-1}) \tag{3.2}$$

The spectrum in the ω^2 variable is contained in the interval $0 \leqslant \omega^2 \leqslant 4$. Assume $4/M < \omega^2 < 4$. Then the mapping T_M^{-1}: $u \to 2 - M\omega^2 - u^{-1}$ has two real fixed points, namely, $u = u_0$ and u_0^{-1}, with

$$u_0 = 1 - \tfrac{1}{2}M\omega^2 + \tfrac{1}{2}[M\omega^2(M\omega^2-4)]^{1/2} \tag{3.3}$$

u_0 satisfies $-1 < u_0 < 0$. If $(1-p)\, u_0^{-2} < 1$, we can solve $W'(u_0)$ from Eq. (3.2). Since W' is a density and cannot be negative, a difficulty arises if the parameters are such that $(1-p)\, u_0^{-2} > 1$. This happens for instance if p is small enough. There are only two ways out: either $W'(u_0)$ is infinite or it is zero. If $W'(u_0)$ is infinite, Eq. (3.2) implies that W' is also infinite at the point $u_1 = (2-\omega^2-u_0)^{-1} \equiv T_1(u_0)$, and by induction that W' is infinite at all the points $u_n = T_1^n(u_0)$ $(n \geqslant 0)$. This set of points is *dense* on the whole real axis if $\omega^2 = 2\ (1-\cos\beta)$ with β/π irrational. Indeed the change of variable

$$w = \frac{u - e^{i\beta}}{u - e^{-i\beta}} \tag{3.4}$$

maps the real axis onto the unit circle and conjugates T_1 with $\tilde{T}_1$: $w \to e^{2i\beta}w$. On the other hand, if $W'(u_0) = 0$, then W' also vanishes at the point $u_{-1} = 2 - \omega^2 - u_0^{-1} = T_1^{-1}(u_0)$, and by induction one finds that $W' = 0$ at $u_{-n} = T_1^{-n}(u_0)$ $(n \geqslant 0)$, which also form a dense set of the real line whenever β/π is irrational.

We now examine in more detail the behavior of $W(u)$ around $u=u_0$. Let $u=u_0+x$. The expansion of Eq. (3.1) up to first order in x reads

$$W(u_0+x)-(1-p)\,W(u_0+\mu x)$$
$$=pW(2-\omega^2-u_0^{-1})-1-W(-\infty)+\cdots \tag{3.5}$$

with

$$\mu=\frac{dT_M^{-1}}{du}(u_0)=u_0^{-2}>1 \tag{3.6}$$

If $W(u_0+x)$ has a *singular* part W_{sg} on top of a smooth background for small x, then this function W_{sg} satisfies the *homogeneous* equation obtained by setting the right-hand side of Eq. (3.5) equal to zero. The general solution of the homogeneous equation reads

$$W_{sg}(u_0+x)=\pm|x|^\alpha P_\pm\left(\frac{\ln|x|}{\ln\mu}\right) \tag{3.7}$$

where the exponent α is given by

$$\alpha=-\frac{\ln(1-p)}{\ln\mu}>0 \tag{3.8}$$

and $P_\pm$ are two arbitrary periodic functions of their argument, with unit period. The subscript $\pm$ refers to the sign of x. This approach predicts the *value* of the exponent α, the existence of periodic amplitudes $P_\pm$, as well as their period, but it is a far more difficult task to determine the form of $P_\pm$. The very same situation occurs in the study of "hierarchical" models which admit an exact renormalization group transform.[12,13]

We have seen that the density $W'(u)$ must be zero or infinite on a dense set of points if $(1-p)\,u_0^{-2}>1$. This condition is equivalent to $\alpha<1$, and clearly W' diverges on a dense set of points. Whenever $\alpha>1$, the density $W'(u_0)$ is entirely determined by the regular part of W, and is therefore regular. When $\alpha<1$, the behavior of the function $W(u_0+x)$ is *dominated* by its singular part as $x\to 0$. In analogy with Schmidt's discussion, one can show by induction that a singular part of the form (3.7) exists around a dense set of points, including $u_n=T_1^n(u_0)$ $(n\geqslant 0)$.

Let us now give an intuitive argument for the existence of this set of singularities. Assume that p is small and consider a frequency outside the spectrum of heavy masses: $4/M<\omega^2<4$. In order to perform a small-p expansion of $W(u)$, it is useful to consider W as the distribution function of Z_0 in a semi-infinite chain. Since Z_0 only depends on the masses m_n left of the origin ($n \leqslant 0$), the dominant contribution corresponds to $m_n = M$ for all n, and therefore $Z_0 = u_0$ (the stable fixed point of T_M). The next contribution corresponds to the insertion of one light mass at site $(-K)$. The associated $Z_0^{(K)}$ reads

$$Z_0^{(K)} = T_M^K \circ T_1(u_0) \tag{3.9}$$

In order to evaluate this number in a closed form, let us define v through $M\omega^2 = 2(1 + \cosh v)$ (implying $u_0 = -e^{-v}$) and perform the change of variable:

$$w = \frac{u + e^{-v}}{u + e^{v}} \tag{3.10}$$

which is the analog of Eq. (3.4) for transformations with real fixed points. This conjugates T_M with $\tilde{T}_M$: $w \to e^{-2v} w$. One easily obtains

$$w^{(K)} = e^{-(2K+1)v} \frac{(M-1)(1+e^{-v})}{2M-1-e^{v}} \tag{3.11}$$

Since all the events under consideration ($K = 0, 1, 2,...$) have the same relative probability, they give a logarithmically singular contribution to $W(u)$ at u_0. If one pursues the small-p expansion of $W(u)$ by inserting more than one light mass, this logarithmic divergence will be smoothed out, but not disappear. The reason is that, for $\omega^2 > 4/M$ (i.e., outside the spectrum of the heavy masses), these heavy masses act as damping centers for the excitations of the light particles.

This argument explains why, at least for small p, a singularity in $W(u)$ can be expected at $u = u_0$. The precise form of this singularity is given in Eq. (3.7). We can repeat this heuristic analysis for chains which are the same except the fact that $m_0 = m_{-1} = \cdots = m_{-n} = 1$. This leads to singularities of $W(u)$ at $u_n = T_1^n(u_0)$. In this way a physical interpretation for the occurrence of the values $u = u_n$ in Schmidt's discussion has been given. The present argument (considering insertions of light masses in a chain of heavy ones) has been used extensively (see Ref. 14 and references therein). Halperin has shown that it could cause genuine singularities. Let us finally remark that the points $Z_0^{(K)}$ [see Eqs. (3.9)–(3.11)] accumulate on u_0 from *one* side. There exists a special value of the squared frequency, namely,

$$\omega_0^2 = \frac{4M}{2M-1} \tag{3.12}$$

such that $Z_0^{(K)} < u_0$ for $\omega_0^2 < \omega^2 < 4$ and $Z_0^{(K)} > u_0$ for $4/M < \omega^2 < \omega_0^2$. (The number ω_0 will also play a crucial role in next section; it is the eigenfrequency associated with one single light mass in between two semi-infinite pure chains of heavy masses.) The higher-order terms of our small-p expansion, which correspond to insertions of several light masses, are expected to give rise to the same type of singularity (3.7) on *both* sides of u_0.

We have studied the function $W(u)$ by the following numerical procedure. We enumerate exactly the 2^L possible values of Z_L with the initial condition $Z_0=0$ and the corresponding 2^L probabilities. The memory of standard computers allows for values of L up to 18. The approximation $W^{(L)}$ to W we obtain is extremely accurate (at least for $\omega^2>4/M$): this can be checked for instance by comparing $W^{(14)}$ and $W^{(18)}$. Let us illustrate some of our results. Figures 1–3 show plots of the shifted Schmidt function:

$$V(u;\omega^2)=W(u;\omega^2)-W(-\infty;\omega^2) \tag{3.13}$$

for three values of M and ω^2: $M_{(1)}=3/2$ and $\omega^2_{(1)}=32/9$, $M_{(2)}=2$ and $\omega^2_{(2)}=8/3$, $M_{(3)}=5/2$ and $\omega^2_{(3)}=32/15$, respectively. These values are such that $u_0=-1/3$ and $\mu=9$ in the three cases. The probability p reads $p=1-3^{-1/5}=0.192758...$ implying $\alpha=1/10$. No basic difference can be seen between these plots. They show one outstandingly largest singularity at u_0, surrounded by some log-periodic structure. The nature of these

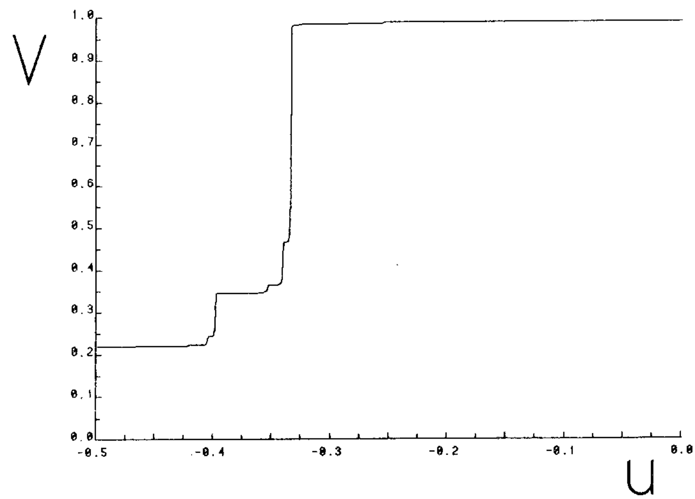

Fig. 1. Plot of the shifted Schmidt function $V(u;\omega^2)$ vs. u, for $M=3/2$; $\omega^2=32/9$; and p such that $\alpha=0.1$.

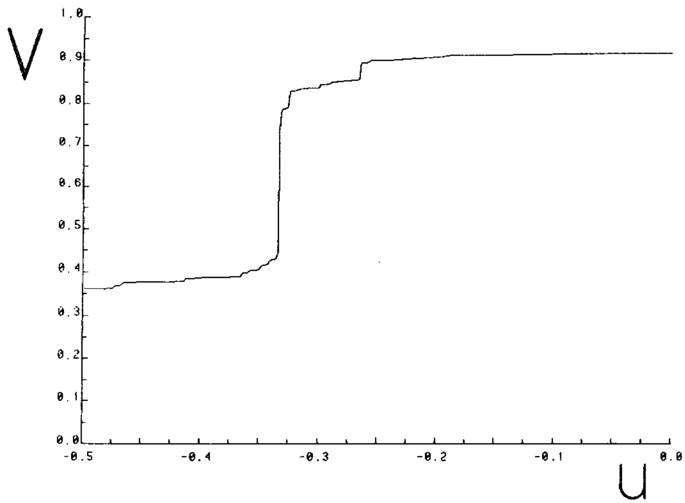

Fig. 2. Same as Fig. 1, for $M=2$; $\omega^2=8/3$.

singularities is the same for $M<2$ as for $M>2$ (see Section 6). Let us recall the definition of the periodic amplitudes $P_\pm$ [see Eq. (3.7)]:

$$\frac{W(u)-W(u_0)}{(u-u_0)^\alpha}=P_+\left[\frac{\ln(u-u_0)}{\ln\mu}\right] \tag{3.14a}$$

$$\frac{W(u_0)-W(u)}{(u_0-u)^\alpha}=P_-\left[\frac{\ln(u_0-u)}{\ln\mu}\right] \tag{3.14b}$$

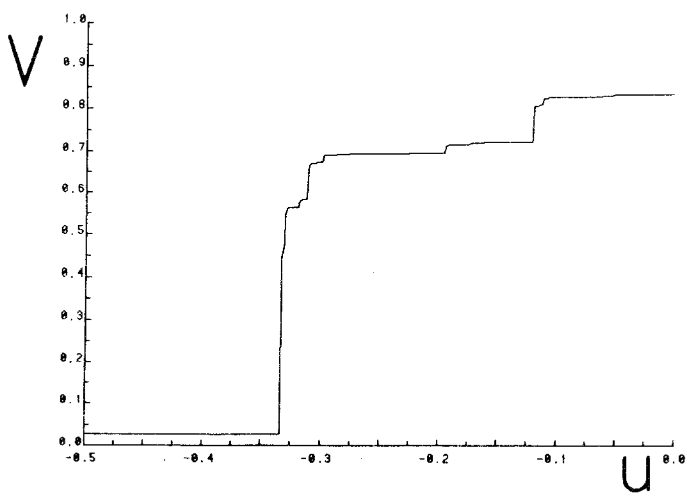

Fig. 3. Same as Fig. 2, for $M=5/2$; $\omega^2=32/15$.

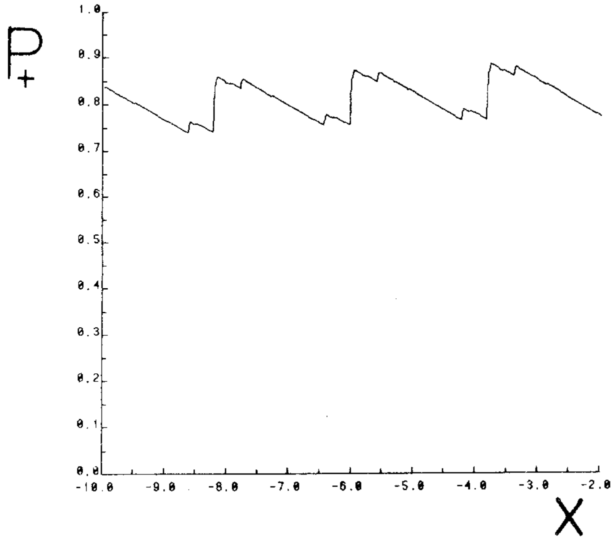

Fig. 4. Plot of the periodic amplitude P_+ vs. $x = \ln(u - u_0)$. The parameters read as in Fig. 3.

It is clear that P_- is much larger than P_+ in Fig. 1, comparable to P_+ in Fig. 2, and much smaller than P_+ in Fig. 3. This is indeed expected from the physical explanation given above, since $\omega_{(1)}^2 > \omega_0^2 = 3$ in the first case, $\omega_{(2)}^2 = \omega_0^2 = 8/3$ in the second one, $\omega_{(3)}^2 < \omega_0^2 = 5/2$ in the third one, and p is small. Figures 4 and 5 show plots of these periodic functions in the most

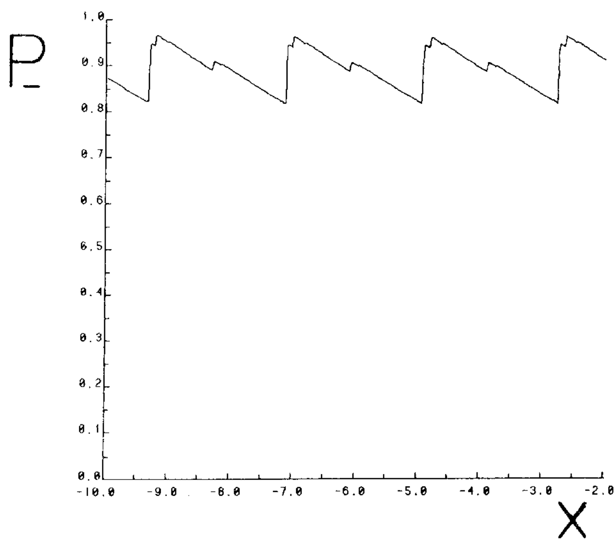

Fig. 5. Plot of the periodic amplitude P_- vs. $x = \ln(u_0 - u)$. The parameters read as in Fig. 1.

favorable cases: P_+ ($M_{(3)}=5/2$, $\omega^2_{(3)}=32/15$) in Fig. 4; P_- ($M_{(1)}=3/2$, $\omega^2_{(1)}=32/9$) in Fig. 5. The abscissa is $x=\ln|u-u_0|$. The period $\ln 9=2.197224...$ is clearly observed.

4. SINGULAR BEHAVIOR OF THE INTEGRATED DENSITY OF STATES

Naively one expects that the singular behavior of the Schmidt function $W(u;\omega^2)$ would cause similar singularities in the integrated density of states $H(\omega^2)$. The latter quantity is given by $H(\omega^2)=-W(-\infty,\omega^2)=V(0,\omega^2)$ [see Eq. (2.7)]. Unfortunately we do not have information on the ω^2 dependence of W at fixed u. In order to exploit our knowledge of the u dependence of W, let us first present a useful relation, which is valid for an *arbitrary* mass distribution $R(m)$ and two arbitrary frequencies ω_1^2, ω_2^2:

$$H(\omega_1^2-H(\omega_2^2)=\iint dR(m)\,dW(u;\omega_1^2) \times [W(2-m\omega_2^2-u;\omega_2^2)-W(2-m\omega_1^2-u;\omega_2^2)] \quad (4.1)$$

This equality can be proven by changing the variable $u\to v=2-m\omega^2-u$ in the second term and eliminating the integrals over $dR(m)$ by use of Eq. (2.6) in the following differential form:

$$dW(u^{-1},\omega^2)=\int dR(m)\,dW(2-m\omega^2-u;\omega^2)$$

Let $\omega_2^2=\omega^2$ be fixed, and $\omega_1^2=\omega^2+\varepsilon$. Assume $\partial W/\partial u$ exists. The limit $\varepsilon\to 0$ then gives

$$H'(\omega^2)=\iint dR(m)\,dW(u;\omega^2)\,m\frac{\partial}{\partial u}W(2-m\omega^2-u;\omega^2) \quad (4.2)$$

when $\partial/\partial u$ in the partial derivative with respect to the first argument. This type of integral equation has been derived by Halperin[15] (see also Ref. 16); Eq. (4.1) is a useful extension of it for situations where $\partial W/\partial u$ is not well behaved.

Let us now show how Eq. (4.1) allows one to predict the behavior of $H(\omega^2)$ for certain values of ω^2, from the known singular behavior of $W(u;\omega^2)$. If $\omega_2^2=\omega^2$, $\omega_1^2=\omega^2+\varepsilon$, and ε is very small, the second argument of the W functions on the right-hand side of Eq. (4.1) can be taken equal to ω^2. (Continuity of $W(u;\omega^2)$ in ω^2 was proven in Section 2.2 of Ref. 16, starting from the Schmidt equation and the known continuity of W in $u^{(2)}$). For the binary mass distribution, we end up with

$$\begin{aligned} &H(\omega^2+\varepsilon)-H(\omega^2) \\ &\quad\approx\int dW(u)\{p[W(2-\omega^2-u)-W(2-\omega^2-\varepsilon-u)] \\ &\qquad+(1-p)[W(2-M\omega^2-u)-W(2-M\omega^2-M\varepsilon-u)]\} \end{aligned} \tag{4.3}$$

The strongest singularity of $H(\omega^2)$ will occur at the value of ω^2 for which both $dW(u)$ and one of the terms in braces simultaneously have their strongest singularity. We have seen in Section 3 that each does so at $u=u_0$, and hence we want to satisfy

$$u_0=2-M\omega^2-u_0 \tag{4.4a}$$

or

$$u_0=2-\omega^2-u_0 \tag{4.4b}$$

The solutions of Eq. (4.4a) are $\omega^2=0$ and $\omega^2=4/M$, i.e., the end points of the spectrum of the heavy masses. We shall study the more interesting end point (Lifshitz) singularity around $\omega^2=4$ in Section 5. The solution of Eq. (4.4b) is ω_0^2, defined already in Eq. (3.12). Let us insert into Eq. (4.3) the singular part of $W(u)$ around $u=u_0$. Set $u=u_0+\eta$. We then have

$$W(u_0+\eta)-W(u_0)\approx|\eta|^\alpha P_\pm(\ln|\eta|/\ln\mu)$$

Consider first the contribution $\delta H_1(\omega_0^2+\varepsilon)$ of the P_-^2 terms for $\varepsilon>0$. It reads

$$\delta H_1(\omega_0^2+\varepsilon)\approx\int_{-\varepsilon<\eta<0}(\varepsilon+\eta)^\alpha P_-\left[\frac{\ln(-\eta)}{\ln\mu}\right]d\left\{(-\eta)^\alpha P_-\left[\frac{\ln(-\eta)}{\ln\mu}\right]\right\} \tag{4.5}$$

It is easy to convince oneself that this expression behaves like

$$\delta H_1(\omega_0^2+\varepsilon)\approx\varepsilon^{2\alpha}R_1\left(\frac{\ln\varepsilon}{\ln\mu}\right)\qquad(\varepsilon>0)$$

where R_1 is another function with period unity. The terms P_+^2, P_+P_-, and P_-^2 give analogous singular contributions to $H(\omega_0^2+\varepsilon)-H(\omega_0^2)$, both for $\varepsilon>0$ and $\varepsilon<0$. Our final result reads

$$H(\omega^2)-H(\omega_0^2)\underset{\omega^2\to\omega_0^2}{\approx}\pm|\omega^2-\omega_0^2|^{2\alpha}R_\pm\left[\frac{\ln|\omega^2-\omega_0^2|}{\ln\mu}\right] \tag{4.6}$$

where $R_\pm$ are two periodic functions with unit period and where α and μ are as in Eqs. (3.6)–(3.8). The functions $R_\pm$ have the form

$$R_\pm(y) = pI_\pm(u_0, u_0, y) \tag{4.7a}$$

where $I_\pm(u_a, u_b, y)$ are integrals involving the periodic functions $P^a_\pm$ and $P^b_\pm$ of $W(u)$ around points u_a and u_b:

$$\begin{aligned} I_\pm(u_a, u_b, y) = &\int_0^1 (1-x)^\alpha P^a_\mp[y + {}^\mu\log(1-x)]\, d[x^\alpha P^b_\mp(y + {}^\mu\log x)] \\ &+ \int_0^\infty \{(x+1)^\alpha P^a_\mp[y + {}^\mu\log(x+1)] - x^\alpha P^a_\mp(y + {}^\mu\log x)\} \\ &\times d[x^\alpha P^b_\pm(y + {}^\mu\log x)] \\ &+ \int_0^\infty \{(x+1)^\alpha P^b_\mp[y + {}^\mu\log(x+1)] - x^\alpha P^b_\mp(y + {}^\mu\log x)\} \\ &\times d[x^\alpha P^a_\pm(y + {}^\mu\log x)] \end{aligned} \tag{4.7b}$$

where ${}^\mu\log x \equiv \ln x/\ln \mu$. Apart from the convolution of the singular parts of the two W functions, there are also convolutions of singular with regular components, behaving like $|\varepsilon|^{1+\alpha}$, as well as convolutions of the regular parts. As these contributions do not yield divergencies in the spectral density, we will not discuss them further. Note, however, that the $|\varepsilon|^{1+\alpha}$ terms, and not the $|\varepsilon|^{2\alpha}$ terms discussed above, are the leading singular terms when $\alpha > 1$.

In previous section we have presented a physical argument explaining why the Schmidt function can be expected to be singular at the fixed point u_0 of T_M, and at the points $u_n = T_1^n(u_0)$. The same picture can be extended to the integrated density of states. Consider an infinite chain of heavy masses, with only one light mass at the origin. The fixed boundary conditions imply the following relation [see Section 2]:

$$T_M^\infty \circ T_1 \circ T_M^\infty(0) = \infty \tag{4.8}$$

In other words, the stable fixed point u_0 of T_M and the unstable one u_0^{-1} have to satisfy:

$$T_1(u_0) = u_0^{-1} \tag{4.9}$$

This relation is precisely equivalent to Eq. (4.4b), and hence leads to $\omega^2 = \omega_0^2$. $H(\omega^2)$ is expected to be singular at that point because many eigenfrequencies around ω_0^2 occur in chains which have $m_0 = 1$ and $m_{\pm 1} =$

Table I. Contributions to the Periodic Functions $R_\pm$ Arising from Convolutions of Singular Parts of $W(u)$ in Eq. (4.3) Around $u=u_a$, $2-\omega^2-u=u_b$ or $u=u_a$, $2-M\omega^2-u=u_c$.

Island	u_a	u_b	u_c	$R_\pm(y)$
$H^\infty LH^\infty$	u_0	u_0	—	$pI_\pm(u_0, u_0; y)$[a]
$H^\infty L^2H^\infty$	u_0	Lu_0	—	$pI_\pm(u_0, Lu_0; y)$
	Lu_0	u_0	—	$+pI_\pm(Lu_0, u_0; y)$
$H^\infty L^3H^\infty$	u_0	L^2u_0	—	$pI_\pm(u_0, L^2u_0; y)$
	Lu_0	Lu_0	—	$+pI_\pm(Lu_0, Lu_0; y)$
	L^2u_0	u_0	—	$+pI_\pm(L^2u_0, u_0; y)$
$H^\infty LHLH^\infty$	u_0	HLu_0	—	$pI_\pm(u_0, HLu_0; y)$
	Lu_0	—	Lu_0	$+(1-p)M^{2\alpha}I_\pm(Lu_0, Lu_0; y+{}^{\mu}\log M)$
	HLu_0	u_0	—	$+pI_\pm(HLu_0, u_0; y)$
$H^\infty L^2HLH^\infty$ and $H^\infty LHL^2H^\infty$[b]	u_0	$LHLu_0$	—	$2pI_\pm(u_0, LHLu_0; y)$
	$LHLu_0$	u_0	—	
	Lu_0	HLu_0	—	$+2pI_\pm(Lu_0, HLu_0; y)$
	HLu_0	Lu_0	—	
	L^2u_0	—	Lu_0	$+2(1-p)M^{2\alpha}I_\pm(L^2u_0, Lu_0; y+{}^{\mu}\log M)$
	Lu_0	—	L^2u_0	
	HL^2u_0	u_0	—	$+2pI_\pm(HL^2u_0, u_0; y)$
	u_0	HL^2u_0	—	

[a] The $I_\pm$ are defined in Eq. (4.7b). Note that $I_\pm(u_a, u_b; y) = I_\pm(u_b, u_a; y)$.
[b] Note that the islands $H^\infty LHL^2H^\infty$ and $H^\infty L^2HLH^\infty$ have the same eigenfrequencies.

$m_{\pm 2} = \cdots = m_{\pm l} = M$ for instance, since the heavy masses damp the excitations of the light ones far away from the origin for $\omega^2 > 4/M$, as discussed in Section 3. From this argument it is clear that singularities are also expected at frequencies associated with sequences of the form *HHHHLLHHHH*; *HHHLHLHHHH*; *HHHHLLLHHHHH*, etc. These are determined by the equations: $T_1^2(u_0) = u_0^{-1}$; $T_1 T_M T_1(u_0) = u_0^{-1}$; $T_1^3(u_0) = u_0^{-1}$ respectively. These relations generalize Eq. (4.4b) to every case where the two functions W occurring in Eq. (4.1) are simultaneously singular. Sequences with a few light masses in a heavy background have been discussed extensively in the literature (see [14] for a review). For islands with two or more light particles, there are several convolutions of singular parts of the Schmidt function which contribute in Eq. (4.3). Consider the contribution of the singularities of the W functions at $u = u_a$, $2 - \omega^2 - u_a = u_b$ or at $u = u_a$, $2 - M\omega^2 - u_a = u_c$. Let us take the example of the island $H^\infty L^2 HLH^\infty$. Its eigenfrequencies are determined by $L^2HLu_0 = u_0^{-1}$ (where the letters L, H stand for T_1, T_M, respectively). One easily verifies that this is equivalent to $LHLu_0 = (Lu_0)^{-1}$; other equivalent forms are $HLu_0 = (L^2u_0)^{-1}$, $Lu_0 = (HL^2u_0)^{-1}$, $u_0 = (LHL^2u_0)^{-1}$. These expressions have the physical interpretation of cutting the island somewhere and equating the continued fractions to the left and to the right. The contributions to Eq. (4.3) corresponding to these relations arise from convolutions of singular parts in different regions of the interval of integration. The table presents the results for the first few islands. Note that, if there is a contribution from $u = u_a$, $2 - \omega^2 - u_b = u_a$, there is also one from $u = u_b$, $2 - \omega^2 - u_b = u_a$. This is related to the fact that the eigenfrequencies of these islands in a sea of heavy masses do not change if the order of the masses in the island is reversed. Also related is the symmetry $I_\pm(u_a, u_b, y) = I_\pm(u_b, u_a, y)$. For the terms involving $u_c = 2 - M\omega^2 - u_a$, one has to keep in mind that ε is replaced by $M\varepsilon$.

We now show that the set of frequencies where the singularity (4.7) occurs is *dense* in the interval $4/M < \omega^2 < 4$. Indeed consider the sequence ω_n^2 defined by

$$T_1^n(u_0) = u_0^{-1} \tag{4.10}$$

By using the conjugation of T_1 with $\tilde{T}_1: w \to e^{2i\beta}w$, with w as in Eq. (3.4) and $\omega^2 = 2(1 - \cos\beta)$, we transform (4.10) into

$$e^{2in\beta} = \frac{2 - u_0^{-1}e^{-i\beta} - u_0 e^{i\beta}}{2 - u_0^{-1}e^{i\beta} - u_0 e^{-i\beta}} = e^{i\varphi(\beta)} \tag{4.11}$$

Since $(e^{2in\beta})_{n \geqslant 0}$ is dense in the unit circle whenever β/π is irrational, and $\varphi(\beta)$ is a smooth function of β, Eq. (4.11) is satisfied on a dense set of values of ω, provided $\varphi(\beta)$ is real, i.e., $\omega^2 > 4/M$, which proves our claim. Thus, for $4/M < \omega^2 < 4$, $dH/d\omega^2$ has singularities. This density of states, however, *diverges* only when $\alpha < 1/2$, i.e., in the subinterval $[\omega_-^2; 4]$ with

$$\omega_-^2 = \frac{(2-p)^2}{M(1-p)} \tag{4.12}$$

This interval only exists for

$$M > 1 + \frac{p^2}{4(1-p)} \tag{4.13}$$

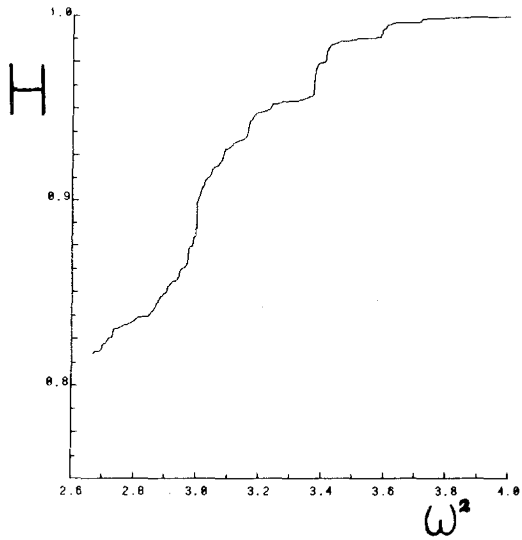

Fig. 6. Plot of the integrated density of states $H(\omega^2)$, on the whole singular interval $4/M < \omega^2 < 4$, for $p = 1/4$ and $M = 3/2$.

Just as for the Schmidt function $W(u)$, we have studied the integrated density of states $H(\omega^2)$ numerically, by exact enumeration up to a chain size $L = 18$. The variations of $H(\omega^2)$ in the whole interval $4/M < \omega^2 < 4$ where it is singular are plotted on Figs. 6–9 for $p = 1/4$ and $M = 1.5$, $M = 2$, $M = 3$ and $M = \infty$ respectively. The last value ($M = \infty$) corresponds to an

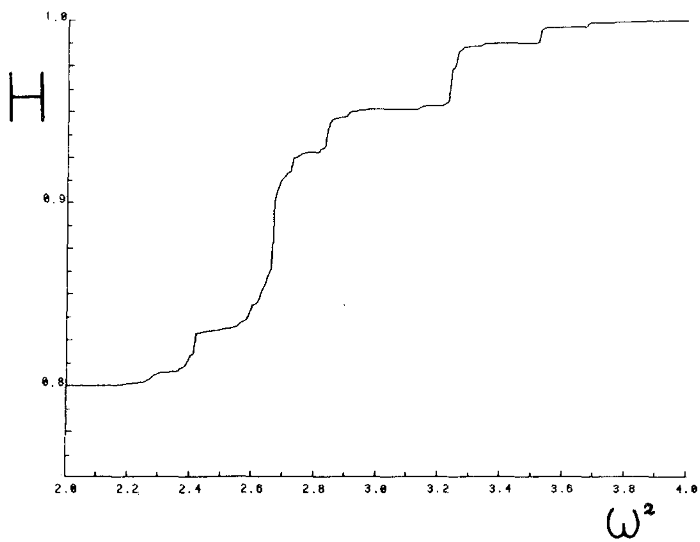

Fig. 7. Same as Fig. 6, for $M = 2$.

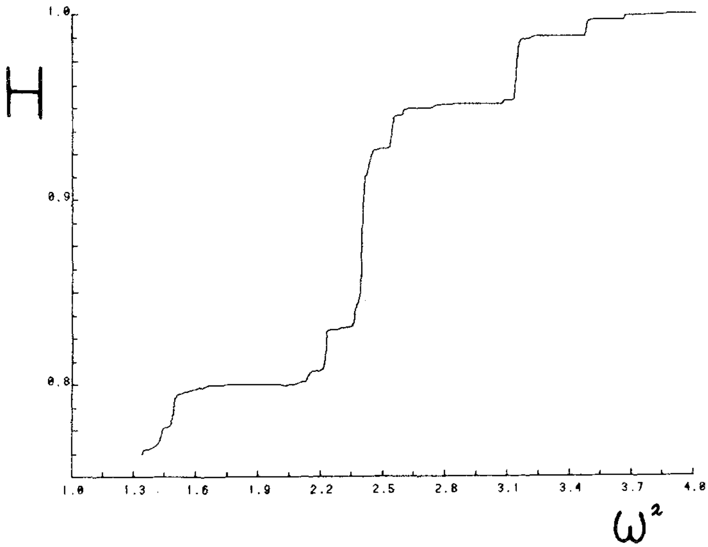

Fig. 8. Same as Fig. 6, for $M = 3$.

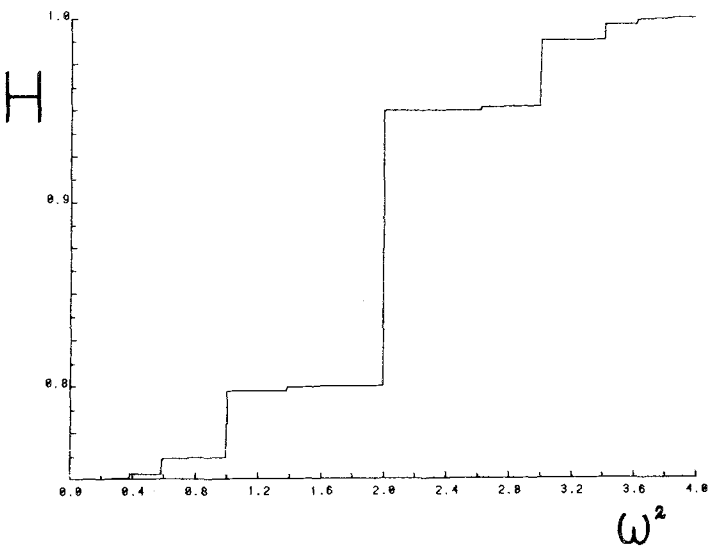

Fig. 9. Same as Fig. 6, for $M \to \infty$.

exactly soluble case[17] which will be studied in detail in Section 5. The two largest singularities always occur at $\omega^2 = \omega_0^2$ [see Eq. (3.12)], solution of $T_1(u_0) = u_0^{-1}$, and $\omega^2 = \omega_1^2$, with

$$\omega_1^2 = \frac{3M - 4 + [M(9M-8)]^{1/2}}{2(M-1)} \tag{4.14}$$

solution of $T_1^2(u_0) = u_0^{-1}$. Our discussion of insertions of one or two light masses in a pure background of heavy masses is therefore fully relevant. The periodic amplitudes R_+ and R_- of $H(\omega^2)$ around ω_1^2 are plotted on Figs. 10 and 11, respectively, for $M=4$, and $p = 0.207950...$ such that $\alpha = 1/20$. The periodicity $\mu = u_0^{-2} = 105.9150...$ (since $u_0 = 3 - \omega_1^2$) is clearly observed, although the *smallness* of these quantities ($R_+ \approx 3 \times 10^{-2}$ and $R_- \approx 5 \times 10^{-3}$) prevents our enumeration method from having a good convergence in a wide domain of $x = \ln |\omega^2 - \omega_1^2|$. The convergence of the periodic functions around $\omega^2 = \omega_0^2$ is not good enough to deserve to be presented here.

Figure 12 shows a plot of $H(\omega^2)$ on the whole spectrum $0 < \omega^2 < 4$, for $M = 5$ and $p = 0.5$. As expected, we find a very smooth function on the interval $0 < \omega^2 < \omega_-^2$, and a singular one for $\omega_-^2 < \omega^2 < 4$. The remarkable values of ω^2 ($4/M$, ω_-^2, ω_0^2, ω_1^2) are indicated by arrows.

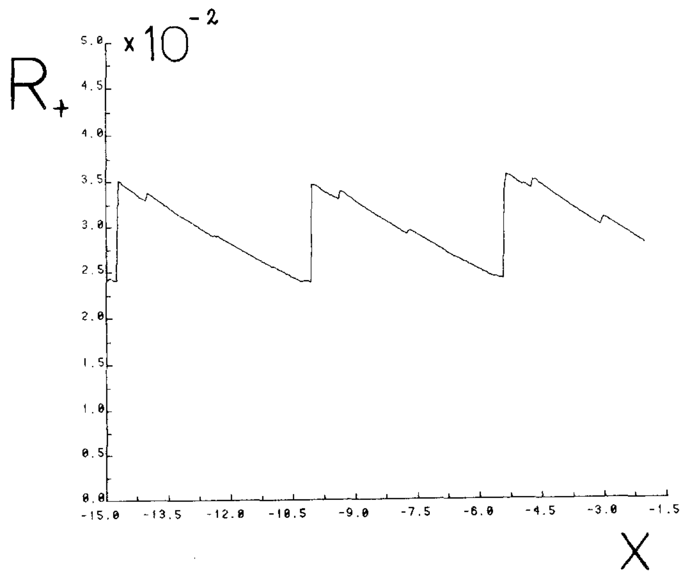

Fig. 10. Plot of the periodic amplitude R_+ of the density of states around ω_1^2, vs. $x = \ln(\omega^2 - \omega_1^2)$, for $M = 4$ and p such that $\alpha = 0.05$.

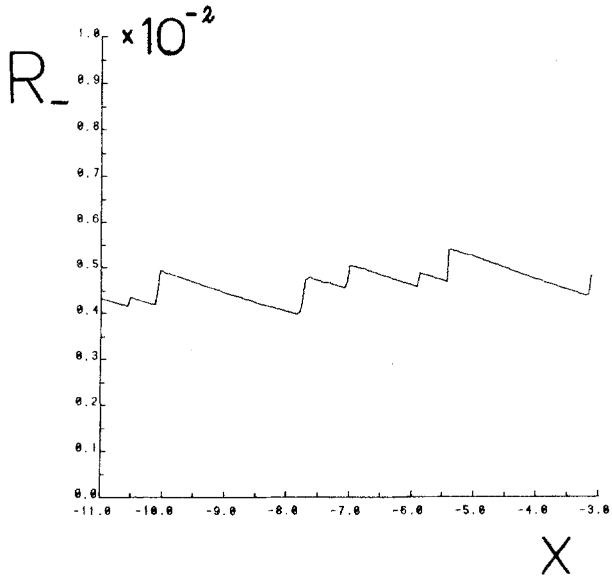

Fig. 11. Plot of the periodic amplitude R_- vs. $x = \ln(\omega_1^2 - \omega^2)$. The parameters read as in Fig. 10.

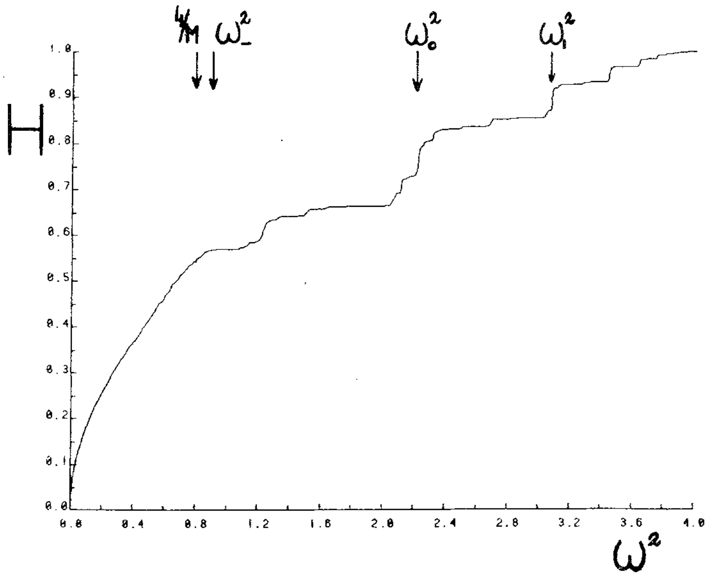

Fig. 12. Plot of the integrated density of states $H(\omega^2)$ for $M = 5$ and $p = 0.5$. Arrows indicate remarkable values of ω^2.

5. THE LIFSHITZ SINGULARITY

This section is devoted to the behavior of $H(\omega^2)$ at $\omega^2 \to 4^-$. A Lifshitz singularity[18] of the type

$$1 - H(\omega^2) \sim \exp[-C(4-\omega^2)^{-1/2}] \tag{5.1}$$

is expected at that point, which is the band edge of a pure medium containing only light masses, and hence the band edge of the random model under consideration. See Ref. 35 for a recent review.

Let us first investigate this Lifshitz singularity in the limit where the mass ratio M becomes infinitely large. An exact solution of this case was given first by Domb *et al.*[17] The problem becomes very simple in this limit, since the variable Z_n [see Eq. (2.2)] vanishes whenever $m_n \to \infty$. If we choose the boundary condition $Z_0 = 0$, then it is easy to realize that the variable Z_m assumes only the values $t_l = T_1^l(0)$ with $T_1(Z) = (2 - \omega^2 - Z)^{-1}$ with probabilities

$$q_{m,l} = \text{Prob}\{Z_m = t_l\} = \left| \begin{array}{ll} p^l(1-p), & 0 \leqslant l \leqslant m-1 \\ p^m, & l = m \end{array} \right. \tag{5.2}$$

The limit function $V(u; \omega^2)$ [see Eq. (3.13)] therefore reads

$$V(u) = \sum_{l \geqslant 0} q_{\infty,l} \Theta(u - t_l) \tag{5.3}$$

with $q_{\infty,l} = p^l(1-p)$. The change of variable (3.4) leads easily to the following expression for t_l:

$$t_l = -\frac{\sin l\gamma}{\sin(l+1)\gamma} \tag{5.4}$$

where

$$\omega^2 = 2(1 + \cos\gamma), \qquad 0 \leqslant \gamma \leqslant \pi \tag{5.5}$$

In other terms, $\gamma = \pi - \beta$ with β as in Eq. (3.4). The integrated density of states is then given by Eq. (2.7), namely,

$$H(\omega^2) = (1-p) \sum_{l \geqslant 0} p^l \Theta(-t_l) \tag{5.6a}$$

or

$$1 - H(\omega^2) = (1-p) \sum_{l \geqslant 0} p^l \Theta(t_l) \tag{5.6b}$$

It is easy to check that $\Theta(t_l)$ is nonzero if and only if there exists an integer $k \geqslant 1$, such that

$$l\gamma < k\pi < (l+1)\gamma \tag{5.7}$$

and therefore one obtains

$$1 - H(\omega^2) = (1-p) \sum_{k \geqslant 1} p^{[kx]} \tag{5.8}$$

where square brackets denote the *integer* part, and where the variable x reads

$$x = \frac{\pi}{\gamma} \tag{5.9}$$

When ω^2 varies from 0 to 4, γ decreases from π to 0, and x increases from 1 to $+\infty$.

Equation (5.8) implies that $H(\omega^2)$ is *discontinuous* at every *rational* value of x. Indeed, if $x = a/b$ where the integers a and b satisfy $1 \leqslant b < a$ and are mutually prime, then H presents a discontinuity $\Delta(\omega^2)$ at the associated value of ω^2, which reads

$$\Delta(\omega^2) = \lim_{\varepsilon \to 0} [H(\omega^2 + \varepsilon) - H(\omega^2 - \varepsilon)] = \frac{(1-p)^2 p^{a-1}}{1-p^a} \tag{5.10}$$

The values of these discontinuities have the following simple interpretation. The factor $(1-p)^2 p^{a-1}$ is the probability for the occurrence of a sequence of $(a-1)$ light masses surrounded by two infinitely heavy ones. Indeed such a subchain has eigenmodes at $\omega^2 = 2[1 + \cos \pi(b/a)]$. The factor $(1-p^a)^{-1}$ is caused by chains with length ja ($j = 2, 3,...$) which also have modes at these frequencies, and a relative probability $p^{(j-1)a}$.[17] The largest discontinuity occurs for $\omega^2 = 2$ ($x = 2$):

$$\Delta(2) = \frac{p(1-p)}{1+p} \tag{5.11}$$

(see Fig. 9).

These discontinuities, which occur only in the $M \to \infty$ limit, can be viewed as the $\alpha \to 0$ limit of the singular behavior (4.7), since α defined in Eq. (3.8) indeed goes to 0 when $M \to \infty$ at fixed ω^2. In particular, the value ω_0^2 [see Eq. (3.12)] goes to 2, where the largest discontinuity (5.11) occurs. This phenomenon is also illustrated by Figs. 6–9.

Let us now turn to the Lifshitz singularity. When ω^2 is close to 4, x is large, and the sum (5.8) is dominated by the term $k = 1$. One has therefore

$$1 - H(\omega^2) \approx p^x Q(x) \tag{5.12}$$

where $Q(x)$ is the following function:

$$Q(x) = (1 - p) \cdot p^{[x] - x} \qquad (M \to \infty) \tag{5.13}$$

This result is indeed of the type (5.1), with

$$C = \pi \, |\ln p| \tag{5.14}$$

but the striking feature is that it is modulated by a function $Q(x)$, which is periodic in x, and not in $\ln x$. Q has period unity, just as the fractional part: $x \to x - [x]$.

The behavior (5.12) of $1 - H(\omega^2)$ can be shown to hold for arbitrary values of M, with the definition (5.9) of x, and where Q is some M-dependent amplitude with period unity. The complete argument will be given elsewhere,[32] since it is closely related to the study of $H(\omega^2)$ around "special frequencies" (see next section). Let us just mention that we obtain

$$1 - H(\omega^2) \underset{x \to \infty}{\approx} (1 - p)\, Z(v_0 + x) \tag{5.15}$$

where v_0 is the constant:

$$v_0 = \frac{1}{2}\left[1 - \left(\frac{M}{M-1}\right)^{1/2}\right] \tag{5.16}$$

and the function Z is related to the Schmidt function $W(u; \omega^2)$ at the band edge ($\omega^2 = 4$):

$$Z(x) = W\left(-1 + \frac{1}{x}; 4\right) - W(-1; 4) \tag{5.17}$$

Our result (5.12) is a simple consequence of the fact that Z satisfies the functional equation:

$$Z(x) = pZ(x - 1) \tag{5.18}$$

whenever $x > 1/[4(M - 1)]$.

Figure 13 shows a plot of the periodic amplitude $Q(x)$, for $p = 0.2$ and $M = 3$. Although this function is continuous, it looks very close to the

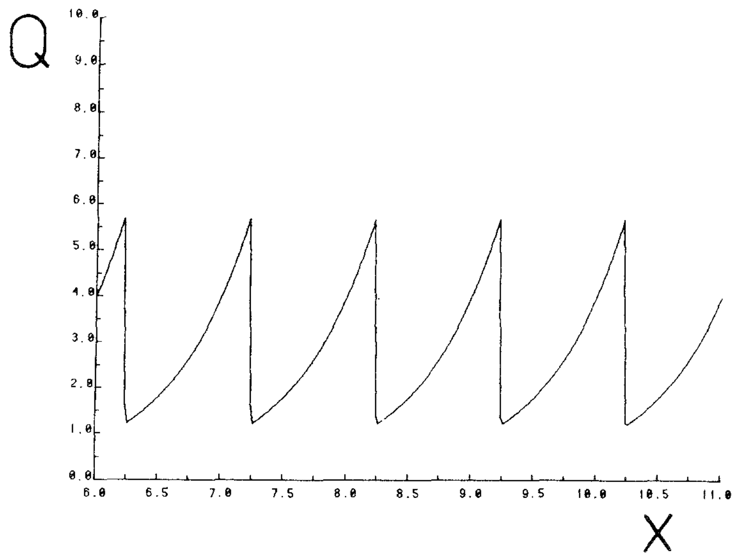

Fig. 13. Plot of the periodic function $Q(x)$ modulating the Lifshitz singularity at $\omega^2=4$, for $M=3$, and $p=0.2$.

$M \to \infty$ result (5.13), which exhibits one discontinuity at each integer. For the values of p and M we have chosen to draw Fig. 13, the asymptotic equality (5.15) is extremely accurate for values of x down to $x=6$.

6. EXTENSION TO OTHER MASS DISTRIBUTIONS; THE LYAPUNOV COEFFICIENT; CONCLUSIONS

We have studied analytically and numerically the Schmidt function $W(u)$ and the integrated density of states $H(\omega^2)$ of a random harmonic chain with a binary mass distribution. Both quantities can be singular on a dense set of points. These power-law singularities of index α are modulated by a log-periodic amplitude with period μ [see Eqs. (3.14) and (4.7)], where both α and μ depend on the location of the singularity.

This phenomenon is caused by the following two combined effects: damping of the excitations of the light particles by the heavy ones (existence of a real u_0), and a "resonance" property (giving rise to a dense set of singular points).

The Lifshitz singularity at the upper edge of the spectrum also exhibits a periodic amplitude.

The singularities in $H(\omega^2)$ cause analogous ones in the Lyapunov coefficient $\gamma(\omega^2)$. Indeed, from the behavior (1.1) of $H(\omega^2)$ and from the formula (2.9) for $\gamma(\omega^2)$, we deduce

$$\gamma(\omega_0^2+\varepsilon)-\gamma(\omega_0^2)\approx|\varepsilon|^{2\alpha}\, T_\pm\left(\frac{\ln|\varepsilon|}{\ln\mu}\right) \tag{6.1}$$

where $T_\pm$ are periodic functions with unit period, given by

$$T_\pm(\log|\varepsilon|)=\int_0^\infty \ln\frac{v+1}{v}\, d[v^{2\alpha}R_\mp({}^\mu\log|\varepsilon|\, v)] + \int_0^\infty \ln\left|\frac{v-1}{v}\right| d[v^{2\alpha}R_\pm({}^\mu\log|\varepsilon|\, v)] \tag{6.2}$$

Note that this relation can also be derived without using Eq. (2.9). Starting from (2.10), we derive, in a similar way to Eq. (4.1),

$$\gamma(\omega_1^2)-\gamma(\omega_2^2)=\iiint dR(m)\, dW(u;\omega_1^2)\, dW(v;\omega_2^2) \times \operatorname{Re}\ln\left(\frac{u+v-2+m\omega_1^2+i0}{u+v-2+m\omega_2^2+i0}\right) \tag{6.3}$$

We set again $\omega_1^2=\omega^2$; $\omega_2^2=\omega^2+\varepsilon$. The points where $u+v-2+\omega^2$ or $u+v-2+M\omega^2$ becomes small and both W functions have a singularity give a contribution proportional to $|\varepsilon|^{2\alpha}$. Collecting all terms, we precisely recover (6.2).

It is interesting to note that Eqs. (4.1) and (6.2) can be combined into one single equation for the difference of $\gamma(\omega^2)+i\pi H(\omega^2)$ at ω_1^2 and ω_2^2. That equation has the form of Eq. (6.3) where the whole logarithm is taken instead of its real part. This is related to the fact that γ and πH are, respectively, the real and imaginary parts of an analytic function, which can be calculated from an integral equation valid for complex values of ω^2.[3]

We have performed a numerical computation of $\gamma(\omega^2)$ for the case $M=10$, $p=0.1$, by our exact enumeration procedure and using Eq. (2.10).

Figure 14 shows a plot of $\gamma(\omega^2)$. One indeed observes deep singularities at ω_0^2 and ω_1^2, defined in Eqs. (3.12) and (4.14), where $H(\omega^2)$ also has its largest singularities. The values $\gamma(\omega_0^2)=0.7$ and $\gamma(\omega_1^2)=2.73$ have been estimated by extrapolation using Eq. (6.1). At the other points where H has a singularity, γ will behave like (6.1), but these "dips" are expected to be much smaller than the two we have discussed, and they are not resolved in our plot.

It has been shown, both using the weak-disorder expansion,[19–21] and in exact solutions,[22] that $\gamma(\omega^2)$ has another dense set of special values of

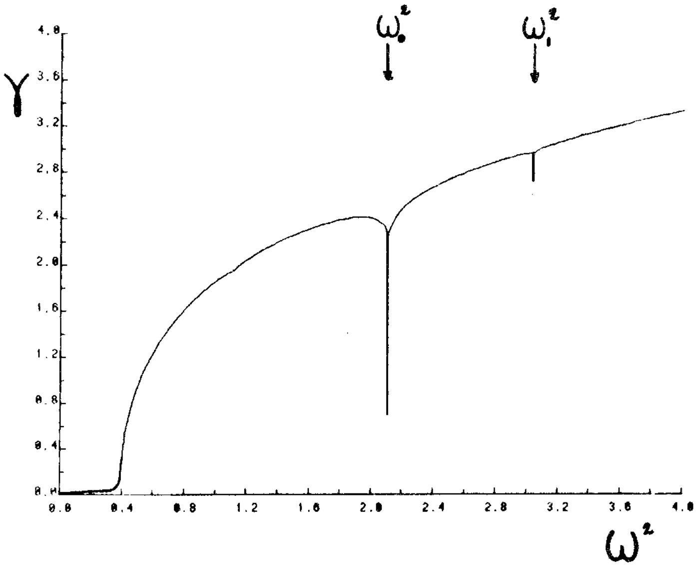

Fig. 14. Plot of the Lyapunov coefficient $\gamma(\omega^2)$ for $M=10$ and $p=0.1$.

ω^2 called anomalies. These are not related to the singularities we discuss here, for at least two reasons: the anomalies do not exhibit power-law behavior, and occur for every kind of mass distribution.

Since the irregular behavior of H occurs only at high frequencies ($4/M<\omega^2<4$), it will not affect thermodynamical quantities. A computation of the derivative of the specific heat with respect to temperature has been recently performed[23] by a method involving the cumulants of the mass distribution. Although some dependence on the kind of distribution is observed, a smooth temperature dependence is also found for binary distributions.

It has been speculated that the singular behavior of the density of states we have considered could be related to the presence of "special frequencies." These are isolated frequencies where the density of states vanishes exponentially fast.[24–26,32] This occurs only if the mass ratio is larger than or equal to 2. Since the singularities we have considered exist for arbitrary mass ratio provided inequality (4.13) is satisfied, also these two phenomena are clearly not related.

Although the present study has been limited to binary mass distributions, our results remain valid for arbitrary *discrete* mass distributions:

$$dR(m) = \sum_{1\leqslant k\leqslant N} P_k \delta(m-m_k)\, dm \tag{6.4}$$

with $N \geqslant 2$, $0 < P_k < 1$, and $m_1 < m_2 < \cdots < m_N$. Each species with mass m_k, except the lightest one ($k = 1$), gives rise to a singular component of $H(\omega^2)$ for $4/m_k < \omega^2 < 4/m_1$, and to divergences of $dH/d\omega^2$ for $\omega_-^2(k) < \omega^2 < 4/m_1$, with

$$\omega_-^2(k) = \frac{(1+P_k)^2}{m_k P_k} \tag{6.5}$$

When the distribution $dR(m)$ contains, apart from a discrete contribution of the type (6.4), also an absolutely continuous component, the situation does not change, as long as the same argument of a dense set of singularities caused by fixed points can be repeated.

In models where the mass distribution $dR(m)$ has a smooth density, one does not expect any singularity in the density of states, except possibly at the band edges. This is indeed verified for exponential and gamma distributions, where the integrated density of states, the localization length and the one-particle Green's function can be obtained exactly.(16) There also exist rigorous bounds on the density of states, given bounds on the density of the mass distribution.(27,36)

The modulation of power-law singularities by log-periodic functions has also been found in other one-dimensional disordered models, such as diffusion in a random medium,(28) or the random field Ising chain.(29) Reference 30 also discusses this problem in a simpler example. This periodic structure only exists if the distribution of the hopping probabilities, magnetic fields, masses,... is a discrete one.

The present work shows that one should be careful in extracting a continuous density of states in binary, ternary, etc., random one-dimensional systems from numerical or other approximate methods, since this quantity may not be well defined under some circumstances. A histogram is more appropriate!

It would be interesting to know whether the measure $H(\omega^2)$ has a *singular continuous* component (in the sense of measure theory). A purely singular continuous measure which has very analogous properties to $H(\omega^2)$ has been studied in the mathematical literature.(31) The singularities we have discussed might also occur in the localization problem.

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